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Integration of stochastic ordinary differential equations from a symmetry standpoint

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Abstract

We derive an algorithm for calculating Lie point symmetries of systems of stochastic ordinary differential equations (SODEs) of any order. From this algorithm, the following facts emerge. Symmetries of a SODE do not in general form a Lie algebra. The determining equations for Ito's equation are in general stochastic linear partial differential equations whereas for SODEs of order $n \geqslant 2$ the determining equations are linear deterministic partial differential equations that form an overdetemined system which is solvable by classical methods.

For scalar second-order SODEs, we provide a complete classification of equations admitting finite-dimensional symmetry Lie algebras. This classification is applied to the integration of scalar second-order SODEs: in general a SODE admitting a two-dimensional symmetry algebra is not integrable by quadratures, although it is reducible to a homogeneous Ito equation. In particular, a scalar second-order SODE admitting a two-dimensional symmetry algebra with connected operators is linearizable. We also characterize integrable scalar second-order SODEs admitting three-dimensional symmetry algebras. Finally we show that a SODE can admit maximally a zero-, one-, two-, three- or four-dimensional Lie algebra.

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1. Introduction

Lie's theory of differential equations (DEs) is one of the most systematic ways for constructing exact solutions of deterministic DEs. It exploits the invariance of the DE under (infinitesimal) transformations (symmetries) that allow integration strategies. Indeed, a majority of deterministic DEs integrable by *ad hoc* means are left unchanged by certain transformations, and tricks for solving them rely mainly on the properties of these special transformations.

In contrast to deterministic DEs, attempts to obtain a symmetry algorithm for stochastic ordinary differential equations (SODEs) similar to Lie's are recent. To the best of our knowledge, seminal contributions in this direction were made by Gaeta and Quintero in their pioneering paper [1] (see also references therein). These authors proposed an algorithm for calculating projectable symmetries of Ito's equation which is the model first-order SODE *par excellence*. They then connected the projective symmetries of Ito's equation to the symmetries of the associated Fokker–Planck equation. From their paper, some fundamental questions remain unanswered: are projectible symmetries the sole type of point symmetry for Ito's equation? Does the set of symmetries of Ito's equation form a Lie algebra? How can symmetries be used in integration? In this paper, beside addressing these and other questions, we obtain an algorithm for calculating the point symmetries of the more general system of SODEs[1]

$$x^{(n)} = f(t, x, \dot{x}, \ldots, x^{(n-1)}) + G(t, x, \dot{x}, \ldots, x^{(n-1)}) \frac{\mathrm{d}B(t)}{\mathrm{d}t} \tag{1}$$

where $x = [x_1, \ldots, x_N]^{\mathrm{T}}$, $f = [f_1, \ldots, f_N]^{\mathrm{T}}$, $G = [G_{ij}]$ is an $N \times M$ matrix, B is an M-dimensional Brownian motion and $x^{(k)} = \mathrm{d}^k x/\mathrm{d}t^k$. Equation (1) is an obvious generalization of Ito's equation. It models a variety of phenomena arising in diverse fields: electrical engineering (LCR circuit driven by thermal noise [2]), structural engineering (buckling of columns with random initial displacement [3]), vibration theory (random vibration of strings and rods [4]), population dynamics and disease control [5], finance (pricing of options [6]), seismology (effect of an earthquake on earthbound structures [7]), fluid mechanics (motion of a lifting surface in response to atmospheric turbulence in a steady flight [8]), telecommunication [9], etc.

For the sake of completeness and clarity of exposition, we begin with a brief introduction to stochastic processes in section 2. In fact this is a pretext to introduce Ito's formula and Oksendal's formula for random time change of Brownian motions. These two formulae are the building blocks for our symmetry algorithm. Section 3 is devoted to the derivation of a symmetry algorithm for (1) and to the study of the structure of the symmetries of (1). Also, we discuss the symmetries of some models that arise in applications. Section 4 focuses on (1) in the case $n = 2$. Namely, we provide a symmetry classification of scalar second-order SODEs possessing one-, two- or three-dimensional symmetry algebras. Section 5 deals with maximal symmetries for scalar second-order SODEs. The last section, section 6, summarizes the results obtained in the previous sections.

Throughout this paper, we assume that the reader is familiar with Lie's algorithm for calculating symmetries of deterministic DEs [10–13] and basic notions of probability theory [14].

2. Stochastic process: Ito's formula, Oksendal's formula

In this section, we assume that the basic concepts of probability theory such as probability space, random variable, characteristic function, expectation, variance and covariance are known.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and I an interval.

A *stochastic process* is an application $X : I \times \Omega \longrightarrow \mathcal{R}^N$ such that for all $t \in I$, $X(t, .)$ is a random variable. For each $t \in I$, the random variable $X(t, .)$ is denoted by $X(t)$.

Let $I = [0, +\infty)$. A *scalar* (standard) *Brownian motion* or *Wiener–Lévy process* is a stochastic process $B(t)$ satisfying the following properties:

[1] The Fokker–Planck (FP) equation for (1) is obtained by rewriting (1) as a system of Nn Ito equations $\mathrm{d}y = d(t, y)\,\mathrm{d}t + S(t, y)\,\mathrm{d}B$ whose FP equation is $P_t + (d_i P)_{,i} - \frac{1}{2}((SS^{\mathrm{T}})_{ij})_{,ij} = 0$, $P(t, y)$ being the probability density function.

(i) $P(B(0) = 0) = 1$;
(ii) for any finite sequence $\{t_i\} \subset I$ with $t_i < t_{i+1}$, the random variables $B(t_{i+1}) - B(t_i)$ are independent;
(iii) for all $t, s \in I$, the probability distribution of $B(t) - B(s)$ is Gaussian with $E(B(t) - B(s)) = 0$ and $E([B(t) - B(s)]^2) = 2D^2|t - s|$, where D is a nonzero constant.

An *N-dimensional Brownian* motion is a stochastic process $B(t) = [B_1(t), \ldots, B_N(t)]^{\mathrm{T}}$, where the B_is are independent scalar Brownian motions.

Wiener [15] proved that the Brownian motion $B(t)$ is nowhere differentiable in the usual sense. The derivative, in the distribution sense, of the Brownian motion $B(t)$ is called *white noise* and is represented formally as $\frac{\mathrm{d}B}{\mathrm{d}t}$.

The *mean square norm* of a random variable R is defined by

$$||R|| = [E(|R|^2)]^{1/2} = \left(\int_{\Omega} |R|^2 \, \mathrm{d}P \right)^{1/2}$$

with $|R| = \sqrt{\sum_{i=1}^{N} R_i^2}$.

Now let $X(t)$ be a stochastic process such that $||X(t)|| < \infty$ for all $t \in [0, a]$, $a > 0$. Consider a subdivision $0 = t_1 < t_2 < \cdots < t_n = a$ of $[0, a]$ and let $\Delta_n = \max_{1 \leqslant k \leqslant n-1}(t_{k+1} - t_k)$. Form the random variable

$$Y_n = \sum_{k=1}^{k=n-1} X(t_k)[B(t_{k+1}) - B(t_k)]. \tag{2}$$

It is straightforward to check that $||Y_n|| < \infty$.

If there is a random variable Y such that

$$\lim_{n \to +\infty, \Delta_n \to 0} ||Y_n - Y|| = 0$$

Y is called the *Ito integral* of $X(t)$ and is denoted by $\int_0^a X(t) \, \mathrm{d}B\,(t)$.

Remark. Note that in (2), the values of $X(t)$ are not taken at arbitrary points in the interval $[t_k, t_{k+1}]$ but at the point t_k.

An *Ito process* is a stochastic process $X(t)$ defined by

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s, X(s)) \, \mathrm{d}s + \int_{t_0}^{t} G(s, X(s)) \, \mathrm{d}B\,(s) \tag{3}$$

or formally

$$\mathrm{d}X\,(t) = f(t, X(t)) \, \mathrm{d}t + G(t, X(t)) \, \mathrm{d}B(t) \tag{4}$$

where $t_0, t \in I$, f is an N vector-valued function, G is an $N \times M$ matrix-valued function, $B(t)$ is an M-dimensional Brownian motion and the second integral in (3) is Ito's integral.

Now we have the tools to state one of the main formulae of this section.

Theorem 1 (Ito's formula [16]). *Let $F : I \times \mathcal{R}^N \longrightarrow \mathcal{R}^Q$ be an application such that $F(t, .) \in \mathcal{C}^2(\mathcal{R}^N, \mathcal{R}^Q)$ and $F(., x) \in \mathcal{C}^1(I, \mathcal{R}^Q)$ for all $(t, x) \in I \times \mathcal{R}^N$. If $X(t)$ is an Ito process, then $F(t, X(t))$ is also an Ito process with*

$$\mathrm{d}F_i\,(t, X(t)) = F_{i,t}(t, X(t)) \, \mathrm{d}t + F_{i,j}(t, X(t)) \, \mathrm{d}X_j + \tfrac{1}{2} F_{i,jk}(t, X(t)) \, \mathrm{d}X_j \, \mathrm{d}X_k \tag{5}$$

where summation over repeated indices is assumed, the indices following the comma refer to partial differentiation and $\mathrm{d}X_j \, \mathrm{d}X_k$ is evaluated using the convention $\mathrm{d}t \, \mathrm{d}B_l = 0$, $\mathrm{d}B_l \, \mathrm{d}B_p = \delta_{lp} \, \mathrm{d}t$.

Proof. See [17, 18]. □

Remark. In stochastic calculus, Ito's formula is the counterpart of the classical chain rule of calculus. This justifies its huge impact in the development of the theory of stochastic DEs.

Example. It is obvious that the scalar Brownian motion $B(t)$ is an Ito process. Using Ito's formula we find that

$$\begin{aligned} \mathrm{d}B^2 &= 2B\,\mathrm{d}B + \tfrac{1}{2}2\,\mathrm{d}B\,\mathrm{d}B \\ &= \mathrm{d}t + 2B\,\mathrm{d}B. \end{aligned}$$

This implies that

$$\int_0^t B(s)\,\mathrm{d}B(s) = \tfrac{1}{2}B^2(t) - \tfrac{1}{2}t.$$

From this simple example, one can readily appreciate the difference between Ito's integral and the Lebesgue integral.

Theorem 2 (Random time change in Brownian motion [18]). *Let $\alpha(t)$ be a scalar stochastic process satisfying:*

(i) $\alpha(0) = 0$, $\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} > 0$,
(ii) there is a stochastic process $\beta(t)$ such that $\alpha(\beta(t,\omega),\omega) = \beta(\alpha(t,\omega),\omega) = t$ for all $(t,\omega) \in I \times \Omega$.

Then, under the (random) time change $\bar{t} = \alpha(t)$, the Brownian motion $B(t)$ is mapped to another Brownian motion $\bar{B}(\bar{t})$ defined by

$$\mathrm{d}\bar{B} = \sqrt{\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}}\,\mathrm{d}B. \tag{6}$$

Proof. Consult [18]. □

Remark. A particular instance of Oksendal's formula was proved in the appendix of [1]. We suspect that the authors were unaware of Oksendal's result.

Example. Using (6), it is straightforward to see that Brownian motions are invariant under time translations.

3. Symmetry algorithm for SODEs

We construct a symmetry algorithm for calculating symmetries of SODEs of any order. We start by recalling the prolongation formulae and some of their properties.

3.1. Prolongation formulae

Consider a one-parameter group of point transformations

$$\bar{t} = f(t, x; \epsilon) \qquad \bar{x}_i = \psi_i(t, x; \epsilon) \qquad i = 1, \ldots, N \tag{7}$$

where ϵ is a small parameter, t is the independent variable and x the dependent variable. It is well known from Lie's work that the group whose transformations are given by (7) can equivalently be described by the infinitesimal transformations

$$\bar{t} = t + \epsilon\xi(t, x) \qquad \bar{x}_i = x_i + \epsilon\eta_i(t, x) \tag{8}$$

where

$$\xi = \left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} \qquad \eta_i = \left.\frac{\partial \psi_i}{\partial \epsilon}\right|_{\epsilon=0}$$

if we assume that the identity is given by $\epsilon = 0$. The *symbol* or *generator* of the group given by (7) is the vector field

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta_i(t, x)\frac{\partial}{\partial x_i}. \tag{9}$$

We are interested in how $\frac{\mathrm{d}^k x_i}{\mathrm{d}t^k} \equiv x_i^{(k)}$ transforms when (7) acts on a DE. The answer is provided by the *prolongation formulae*:

$$\overline{x_i^{(k)}} = x_i^{(k)} + \epsilon\eta_i^{[k]}(t, x, \ldots, x^{(k)}) \qquad k \geqslant 1 \tag{10}$$

where $\eta_i^{[k]}$ is defined recursively by

$$\eta_i^{[k]} = \frac{\mathrm{d}\eta_i^{[k-1]}}{\mathrm{d}t} - x_i^{(k)}\frac{\mathrm{d}\xi}{\mathrm{d}t} \qquad \eta_i^{[0]} = \eta_i. \tag{11}$$

The kth prolongation of the vector field X is given by

$$X^{[k]} = X^{[k-1]} + \eta_i^{[k]}\frac{\partial}{\partial x_i^{(k)}} \qquad X^{[0]} = X. \tag{12}$$

Next we give a lemma that will be useful in the derivation of our symmetry algorithm.

Lemma 1. *For $p \geqslant 1$,*

$$\frac{\partial \eta_i^{[p]}}{\partial x_j^{(p)}} = \eta_{i,j} - p\frac{\mathrm{d}\xi}{\mathrm{d}t}\delta_{ij} - \dot{x}_i\xi_{,j} \tag{13}$$

$$\frac{\partial^2 \eta_i^{[p]}}{\partial x_j^{(p)}\partial x_k^{(p)}} = -\delta_{1p}(\delta_{ij}\xi_{,k} + \delta_{ik}\xi_{,j}). \tag{14}$$

Proof. By induction on p. □

3.2. Point symmetries of Ito's equation

Consider an Ito equation

$$\mathrm{d}x_i = f_i(t, x)\,\mathrm{d}t + G_{ij}(t, x)\,\mathrm{d}B_j \tag{15}$$

invariant under the transformations (7). Using Ito's formula we have up to order ϵ

$$\mathrm{d}\bar{x}_i \approx f_i(t, x)\,\mathrm{d}t + G_{ij}(t, x)\,\mathrm{d}B_j + \epsilon[(\eta_{i,t} + \eta_{i,j}f_j + \tfrac{1}{2}\eta_{i,jk}G_{jl}G_{kl})\,\mathrm{d}t + \eta_{i,j}G_{jk}\mathrm{d}B_k]. \tag{16}$$

Since (15) is invariant under (7), we must have

$$d\bar{x}_i = f_i(\bar{t}, \bar{x})\, d\bar{t} + G_{ij}(\bar{t}, \bar{x})\, d\bar{B}_j. \tag{17}$$

However, up to order ϵ, we have the following approximations:

$$f_i(\bar{t}, \bar{x}) \approx f_i + \epsilon X f_i \qquad G_{ij}(\bar{t}, \bar{x}) \approx G_{ij} + \epsilon X G_{ij} \tag{18}$$

$$d\bar{t} \approx dt + \epsilon[(\xi_{,t} + \xi_{,j} f_j + \tfrac{1}{2}\xi_{,jk} G_{jl} G_{kl})\, dt + \xi_{,j} G_{jk}\, dB_k,]. \tag{19}$$

By using the formula for random time change in the Brownian motion (see section 2) and Ito's formula, we find that

$$d\bar{B}_i \approx dB_i + \frac{\epsilon}{2}\left[\xi_{,t} + \xi_{,j} f_j + \frac{1}{2}\xi_{,jk} G_{jl} G_{kl} + \xi_{,j} G_{jk} \frac{dB_k}{dt}\right] dB_i. \tag{20}$$

Substitute (18)–(20) into (17) and compare the result with (16) to obtain

$$X f_i + f_i(\xi_{,t} + \xi_{,j} f_j + \tfrac{1}{2}\xi_{,jk} G_{jl} G_{kl}) = \eta_{i,t} + \eta_{i,j} f_j + \tfrac{1}{2}\eta_{i,jk} G_{jl} G_{kl}$$

$$X G_{ij} + \frac{1}{2}\left[\xi_{,t} + \xi_{,k} f_k + \frac{1}{2}\xi_{,kl} G_{kp} G_{lp} + \xi_{,k} G_{kl} \frac{dB_l}{dt}\right] G_{ij} + \xi_{,k} G_{kj} f_i = \eta_{i,k} G_{kj}$$

or equivalently (see the appendix for the expanded version of these equations)

$$X^{[1]}(\dot{x}_i - f_i)|_{\dot{x}=f} + \tfrac{1}{2} G_{jl} G_{kl}(\eta_{i,jk} - f_i \xi_{,jk}) = 0 \tag{21}$$

$$X G_{ij} + (\xi_{,k} f_i - \eta_{i,k}) G_{kj} + \frac{1}{2} G_{ij}\left[\xi_{,t} + \xi_{,k} f_k + \frac{1}{2}\xi_{,kl} G_{kp} G_{lp} + \xi_{,k} G_{kl} \frac{dB_l}{dt}\right] = 0. \tag{22}$$

Conversely, if (21) and (22) are satisfied, without loss of generality, we can assume that $X = \partial/\partial t$ (use canonical variables). Thus the equations (21) and (22) will lead to $f_i = f_i(x)$ and $G_{ij} = G_{ij}(x)$, i.e. the Ito equation (15) is invariant under X. So we have proved the following statement.

Theorem 3 (Symmetries of the Ito equation). *A vector field*

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta_i(t, x)\frac{\partial}{\partial x_i}$$

is a symmetry of the Ito equation (15) if and only if (21) and (22) are satisfied.

Remarks. The system (21), (22) is a system of linear stochastic partial DEs. Stochastic because of the *white noise* terms dB_l/dt appearing in (22). So in principle the determining equations for the symmetries of Ito's equation are as difficult to solve as the Ito equation itself. Nevertheless, by making appropriate *antsäze* for ξ and η_i we can simplify the determining equations.

If we assume that X is projectable, i.e. $\xi = \xi(t)$, we recover the algorithm of Gaeta and Quintero [1].

If $G = 0$, we recover the classical algorithm for symmetries of a deterministic equation.

In general, the symmetries of Ito's equation do not form a Lie algebra. Indeed it can be checked that the Lie bracket of two symmetries is not necessarily a symmetry. However, the symmetries of Ito's equation do form a vector field.

3.3. Symmetries of (1) when $n \geqslant 2$

We first rewrite (1) as

$$\mathrm{d}x^{(n-1)} = f(t, x, \dot{x}, \ldots, x^{(n-1)})\,\mathrm{d}t + G(t, x, \dot{x}, \ldots, x^{(n-1)})\,\mathrm{d}B \tag{23}$$

$$\mathrm{d}x^{(k)} = x^{(k+1)}\,\mathrm{d}t \qquad k = 0, 1, \ldots, n-2. \tag{24}$$

Assume that (1) (or equivalently (23), (24)) is invariant under (7). This implies that

$$\mathrm{d}\overline{x^{(n-1)}} = f(\bar{t}, \bar{x}, \bar{\dot{x}}, \ldots, \overline{x^{(n-1)}})\,\mathrm{d}\bar{t} + G(\bar{t}, \bar{x}, \bar{\dot{x}}, \ldots, \overline{x^{(n-1)}})\,\mathrm{d}\bar{B} \tag{25}$$

$$\mathrm{d}\overline{x^{(k)}} = \overline{x^{(k+1)}}\,\mathrm{d}\bar{t} \qquad k = 0, 1, \ldots, n-2. \tag{26}$$

A direct computation using the prolongation formulae and Oksendal's formula for random time change in the Brownian motion shows that

$$\begin{aligned} f_i(\bar{t}, \bar{x}, \ldots, \overline{x^{(n-1)}}) &\approx f_i + \epsilon X^{[n-1]} f_i \\ G_{ij}(\bar{t}, \bar{x}, \ldots, \overline{x^{(n-1)}}) &\approx G_{ij} + \epsilon X^{[n-1]} G_{ij} \\ \mathrm{d}\bar{B}_i &\approx \mathrm{d}B_i + \frac{\epsilon}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}\,\mathrm{d}B_i. \end{aligned} \tag{27}$$

The substitution of (27) into (25) yields

$$\mathrm{d}\overline{x_i^{(n-1)}} \approx f_i\,\mathrm{d}t + G_{ij}\,\mathrm{d}B_i + \epsilon\left[\left(f_i\frac{\mathrm{d}\xi}{\mathrm{d}t} + X^{[n-1]} f_i\right)\mathrm{d}t + \left(\frac{1}{2}G_{ij}\frac{\mathrm{d}\xi}{\mathrm{d}t} + X^{[n-1]}G_{ij}\right)\mathrm{d}B_j\right]. \tag{28}$$

The combined use of the prolongation formulae and Ito's formula gives

$$\begin{aligned} \mathrm{d}\overline{x_i^{(n-1)}} = \mathrm{d}x_i^{(n-1)} + \epsilon\Bigg[\Bigg(\eta_{i,t}^{[n-1]} + \frac{\partial \eta_i}{\partial x_j^{(n-1)}} f_j + \frac{1}{2}\frac{\partial^2 \eta_i}{\partial x_j^{(n-1)}\partial x_k^{(n-1)}} G_{jl}G_{kl} \\ + \sum_{\alpha=0}^{n-2}\frac{\partial \eta_i^{[n-1]}}{\partial x_j^{(\alpha)}} x_j^{(\alpha+1)}\Bigg)\mathrm{d}t + \frac{\partial \eta_i^{[n-1]}}{\partial x_j^{(n-1)}} G_{jk}\,\mathrm{d}B_k\Bigg]. \end{aligned} \tag{29}$$

Comparing (28) with (29), we arrive at

$$\eta_{i,t}^{[n-1]} + \frac{\partial \eta_i}{\partial x_j^{(n-1)}} f_j + \frac{1}{2}\frac{\partial^2 \eta_i}{\partial x_j^{(n-1)}\partial x_k^{(n-1)}} G_{jl}G_{kl} + \sum_{\alpha=0}^{n-2}\frac{\partial \eta_i^{[n-1]}}{\partial x_j^{(\alpha)}} x_j^{(\alpha+1)} = f_i\frac{\mathrm{d}\xi}{\mathrm{d}t} + X^{[n-1]} f_i$$

$$\frac{\partial \eta_i^{[n-1]}}{\partial x_j^{(n-1)}} G_{jk} = \frac{1}{2}G_{ik}\frac{\mathrm{d}\xi}{\mathrm{d}t} + X^{[n-1]}G_{ik}$$

or equivalently

$$X^{[n]}(x_i^{(n)} - f_i)|_{x^{(n)}=f} + \frac{1}{2}\frac{\partial^2 \eta_i}{\partial x_j^{(n-1)}\partial x_k^{(n-1)}} G_{jl}G_{kl} = 0$$

$$X^{[n-1]}G_{ik} + \frac{1}{2}G_{ik}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \frac{\partial \eta_i^{[n-1]}}{\partial x_j^{(n-1)}} G_{jk} = 0.$$

Using lemma 1, we obtain

$$X^{[n]}(x_i^{(n)} - f_i)|_{x^{(n)}=f} - \delta_{1n-1}G_{ik}G_{jk}\xi_{,j} = 0 \tag{30}$$

$$X^{[n-1]}G_{ik} + \left(n - \frac{1}{2}\right)\frac{\mathrm{d}\xi}{\mathrm{d}t}G_{jk} - G_{jk}(\eta_{i,j} - \dot{x}_i\xi_{,j}) = 0. \tag{31}$$

In matrix notation, the system (30), (31) reads (see the appendix for the expansions of these equations)

$$X^{[n]}(x^{(n)} - f)|_{x^{(n)}=f} - \delta_{1n-1}GG^T\,\mathrm{grad}\,\xi = 0 \tag{32}$$

$$X^{[n-1]}G + \left(n - \frac{1}{2}\right)\frac{\mathrm{d}\xi}{\mathrm{d}t}G - (\mathrm{grad}\,\eta)G + (G\dot{x})^T\,\mathrm{grad}\,\xi = 0 \tag{33}$$

where grad $\xi = [\xi_{,i}]^{\mathrm{T}}$, grad $\eta = [\eta_{i,j}]$. Conversely, if X satisfies (32), (33), we can without loss of generality assume that $X = \partial/\partial t$ (use canonical variables), whence $f = f(x, \dot{x}, \ldots, x^{(n-1)})$ and $G = G(x, \dot{x}, \ldots, x^{(n-1)})$, i.e. X is a symmetry of (1). We have thus proved the following result.

Theorem 4. *A vector field $X = \xi(t, x)\frac{\partial}{\partial t} + \eta_i(t, x)\frac{\partial}{\partial x_i}$ is a symmetry of the SODEs (1) with $n \geqslant 2$ if and only if (32) and (33) are satisfied.*

Remarks. The determining equations (32), (33) for SODEs of order $n \geqslant 2$ form an overdetermined system of linear deterministic partial DEs. Recall that for Ito's equation, the determining equations were stochastic.

The symmetries of a SODE of order $n \geqslant 2$ do not in general form a Lie algebra. It can be checked that the Lie bracket of two symmetries is not necessarily a symmetry.

When $G = 0$, we re-obtain Lie's classical symmetry algorithm.

Examples. Consider the response of a mass–spring linear oscillator to a white-noise random excitation. The governing equation is [19, 20]

$$\ddot{x} = -\omega^2 x + \sigma \frac{\mathrm{d}B}{\mathrm{d}t} \tag{34}$$

where $\omega^2 = k/m =$ const., m is the mass, k is the characteristic coefficient of the spring and $\sigma =$ const. $\neq 0$. The determining equations for (34) read

$$X^{[2]}(\ddot{x} + \omega^2 x)|_{\ddot{x}+\omega^2 x=0} - \sigma^2 \eta_{,x} = 0 \tag{35}$$

$$\tfrac{3}{2}(\xi_{,t} + \dot{x}\xi_{,x}) - \sigma\eta_{,x} + \dot{x}\xi_{,x}\sigma = 0. \tag{36}$$

The second determining equation (36) yields

$$\xi = a(t) \qquad \eta = \tfrac{3}{2}\dot{a}(t)x + b(t). \tag{37}$$

Substituting (37) into (35), we obtain, after simple calculations,

$$a = C_1 t + C_2 \qquad \ddot{b} + \omega^2 b = 0 \quad C_1\omega^2 = 0 \tag{38}$$

where C_1 and C_2 are arbitrary constants. The last equation of (38) prompts the consideration of the following cases.

(i) If $\omega = 0$, then $\xi = C_1 t + C_2$, $\eta = \frac{3}{2}C_1 x + C_3 t + C_4$, where the C_i are arbitrary constants. So the symmetries form a vector space spanned by the operators

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = t\frac{\partial}{\partial x} \qquad X_4 = t\frac{\partial}{\partial t} + \frac{3}{2}x\frac{\partial}{\partial x}. \tag{39}$$

Simple computations show that

$$[X_1, X_2] = 0 \qquad [X_2, X_3] = 0 \qquad [X_1, X_3] = X_2$$
$$[X_1, X_4] = X_1 \qquad [X_2, X_4] = \tfrac{3}{2}X_2 \qquad [X_3, X_4] = \tfrac{1}{4}X_3.$$

Thus the symmetries span a Lie algebra.

(ii) If $\omega \neq 0$, then $\xi = C_1$, $\eta = C_2\cos(\omega t) + C_3\sin(\omega t)$, where the C_is are constants. Thus the symmetries are generated by the vectors

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \cos(\omega t)\frac{\partial}{\partial x} \qquad X_3 = \sin(\omega t)\frac{\partial}{\partial x}.$$

These vectors satisfy

$$[X_1, X_2] = -\omega X_3 \qquad [X_1, X_3] = \omega X_1 \qquad [X_1, X_2] = 0.$$

Remarks. The symmetry structure of the mass–spring stochastic oscillator clearly points to the fact that scalar linear SODEs are not equivalent to each other, i.e. they cannot be mapped to each other by means of invertible point (derivative independent) transformations. In contrast note that scalar deterministic linear ODEs are equivalent to the free-particle equation $\ddot{x} = 0$, i.e. they are equivalent to each other. Furthermore, when $\omega = 0$, equation (34) admits a four-dimensional Lie algebra. This is peculiar to SODEs. Indeed, a scalar deterministic ODE cannot admit a four-dimensional Lie algebra [21, 23].

The motion of a hard-spring oscillator driven by a Gaussian white noise is described by [20]

$$\ddot{x} + 2\alpha\dot{x} + \omega^2(x + x^3) = \sigma\frac{\mathrm{d}B}{\mathrm{d}t} \tag{40}$$

where $\alpha, \omega \neq 0$ and $\sigma \neq 0$ are constants. Straightforward calculations reveal that symmetries of (40) are generated by $X = \partial/\partial t$.

4. Classification and integration of scalar second-order SODEs admitting two- or three-dimensional symmetry Lie algebras

We provide a classification of scalar SODEs possessing two- and three-dimensional Lie algebras. Then we show when it is possible to integrate the canonical forms and how this leads to theorems on integrability of scalar second-order SODEs.

4.1. Scalar second-order SODEs with two-dimensional symmetry Lie algebras

There are exactly two classes of two-dimensional Lie algebras, the Abelian two-dimensional Lie algebra $L_2^I : [X_1, X_2] = 0$ and the non-Abelian two-dimensional Lie algebra L_2^{II} : $[X_1, X_2] = X_1$. By assuming the connectedness ($X_1 = \rho(t, x)X_2$) and the unconnectedness ($X_1 \neq \rho(t, x)X_2$) of X_1 and X_2, Lie obtained the nonsimilar realizations of these algebras in terms of vector fields in (1 + 1)-space. He used these realizations to study the integrability of scalar second-order deterministic ODEs. He found that a scalar second-order deterministic ODE admitting a two-dimensional Lie algebra is integrable by quadratures. In order to ascertain whether this assertion is valid for scalar SODEs, we follow Lie's steps. The result of the classification is given in table 1: an equation admitting a given set of symmetries is constructed by imposing these symmetries on the general scalar second-order SODEs via the symmetry algorithm obtained in the previous section. For a given realization of a Lie algebra, the equation obtained is a *representative equation* for that realization since, if that equation is mapped to another equation via an invertible point transformation, the resulting equation enjoys the same symmetry propreties. This is what motivated Lie to look for nonsimilar realizations of Lie algebras in his study of the symmetry structure of deterministic DEs.

A close analysis of table 1 reveals the following theorems.

Theorem 5. *A scalar second-order SODE which admits a two-dimensional point symmetry Lie algebra with connected generators is reducible via an invertible point transformation to a linear scalar second-order SODE.*

Proof. From table 1, we see that the only algebras with connected operators are $L_{2,1}^I$ and $L_{2,1}^{II}$ and the corresponding equations are clearly linear. □

Table 1. Canonical forms of scalar second-order SODEs admitting two-dimensional symmetry Lie algebras.

Algebra	Basis operators	Representative equations
$L^{I}_{2,1}$	$X_1 = \dfrac{\partial}{\partial x}, X_2 = t\dfrac{\partial}{\partial x}$	$\ddot{x} = f(t) + g(t)\dfrac{\mathrm{d}B}{\mathrm{d}t}$
$L^{I}_{2,2}$	$X_1 = \dfrac{\partial}{\partial t}, X_2 = \dfrac{\partial}{\partial x}$	$\ddot{x} = f(\dot{x}) + g(\dot{x})\dfrac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II}_{2,1}$	$X_1 = \dfrac{\partial}{\partial x}, X_2 = x\dfrac{\partial}{\partial x}$	$\ddot{x} = f(t)\dot{x} + g(t)\dot{x}\dfrac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II}_{2,2}$	$X_1 = \dfrac{\partial}{\partial x}, X_2 = t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x}$	$\ddot{x} = t^{-1}f(\dot{x}) + t^{-1/2}g(\dot{x})\dfrac{\mathrm{d}B}{\mathrm{d}t}$

Theorem 6. *A scalar second-order SODE invariant under a two-dimensional point symmetry Lie algebra with unconnected basis vectors is reducible to a homogeneous Ito equation* $\mathrm{d}u = F(u)\,\mathrm{d}\bar{t} + G(u)\,\mathrm{d}\bar{B}$.

Proof. For the equation admitting $L^{I}_{2,2}$, make the change $u = \dot{x}$, $\bar{t} = t$ and for the equation corresponding to $L^{II}_{2,2}$ perform the transformation $u = \dot{x}$, $\bar{t} = \ln t$. □

Remark. Note that a scalar second-order SODE admitting a two-dimensional Lie algebra with unconnected operators is not in general integrable by quadratures.

4.2. Scalar second-order SODEs with three-dimensional symmetry Lie algebras

Lie [21] was the first to completely classify complex three-dimensional Lie algebras. He exploited this classification to obtain complex realizations of three-dimensional Lie algebras in terms of vector fields in (1 + 1)-space. The classification of real three-dimensional Lie algebras is due to Bianchi [22]. The realizations of these algebras in terms of vector fields in (1 + 1)-space were first given in Mahomed [23] (see also [24]). We will use these realizations to obtain canonical forms for scalar SODEs admitting three-dimensional Lie algebras. The results are summarized in table 2. The algebras not appearing in table 2 are those not admissible by a scalar second-order SODE or those leading to deterministic ODEs. In table 2, a, b, α and $\beta \neq 0$ are constants.

It is straightforward to see that linear second-order SODEs admitting three-dimensional Lie algebras are readily integrable. It remains then to study the integrability of the nonlinear ones. We shall need the following theorem.

Theorem 7. *Consider a scalar homogeneous Ito equation*

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}B. \tag{41}$$

Let

$$I = \frac{f}{g} - \frac{1}{2}g' \qquad J = \frac{(gI')'}{I'}. \tag{42}$$

Equation (41) is reducible to a linear SODE via a transformation $u = h(x)$ *if and only if* $I' = 0$ *or* $J' = 0$.

Proof. See Gard [17]. □

Table 2. Canonical forms of scalar second-order SODEs admitting three-dimensional symmetry Lie algebras. Let $p = \partial/\partial t$ and $q = \partial/\partial x$.

Algebras	Basis operators	Representative equations
$L_{3,2}$	$X_1 = q, X_2 = p, X_3 = tq$	$\ddot{x} = \alpha + \beta \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{I}_{3,3}$	$X_1 = q, X_2 = p, X_3 = tp + (t + x)q$	$\ddot{x} = \alpha \mathrm{e}^{-\dot{x}} + \beta \mathrm{e}^{-\dot{x}/2} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II}_{3,3}$	$X_1 = q, X_2 = tq, X_3 = p + xq$	$\ddot{x} = \mathrm{e}^{t} + \mathrm{e}^{t} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II}_{3,4}$	$X_1 = q, X_2 = tq, X_3 = tp + xq$	$\ddot{x} = \alpha t^{-1} + \beta t^{-1/2} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{I,a}_{3,6}$	$X_1 = p, X_2 = q, X_3 = tp + axq$	$\ddot{x} = \alpha \dot{x}^{\frac{a-2}{a-1}} + \beta \dot{x}^{\frac{a-3/2}{a-1}} \frac{\mathrm{d}B}{\mathrm{d}t}, a \neq 0, 1$
$L^{II,a}_{3,6}$	$X_1 = q, X_2 = tq, X_3 = (1 - a)tp + xq$	$\ddot{x} = \alpha t^{\frac{2a-1}{1-a}} + \beta t^{\frac{a-3/2}{1-a}} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{I,b}_{3,7}$	$X_1 = p, X_2 = q, X_3 = (bt + x)p + (bx - t)q$	$\ddot{x} = (\alpha + \beta^2 \dot{x})(1 + \dot{x}^2)^{3/2} \mathrm{e}^{b \tan^{-1} \dot{x}}$ $+\beta(1 + \dot{x}^2)^{5/4} \mathrm{e}^{\frac{1}{2} b \tan^{-1} \dot{x}} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II,b}_{3,7}$	$X_1 = tq, X_2 = q, X_3 = (1 + t^2)p + (x + b)xq$	$\ddot{x} = \alpha(1 + t^2)^{-3/2} \mathrm{e}^{b \tan^{-1} t}$ $+\beta(1 + t^2)^{-1} \mathrm{e}^{b \tan^{-1} t} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{I}_{3,8}$	$X_1 = q, X_2 = tp + xq, X_3 = 2txp + x^2 q$	$\ddot{x} = (\alpha \dot{x}^3 - \frac{1}{2}\dot{x} + \beta^2 \dot{x}^4)t^{-1}$ $+\beta \dot{x}^{5/2} t^{-1/2} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{II}_{3,8}$	$X_1 = q, X_2 = tp + xq, X_3 = 2txp + (x^2 - t^2)q$	$\ddot{x} = [(\dot{x} + \dot{x}^3) + \alpha(1 + \dot{x}^2)^{3/2}$ $+\beta^2 \dot{x}(1 + \dot{x}^2)^{3/2}]t^{-1}$ $+\beta(1 + \dot{x}^2)^{5/4} t^{-1/2} \frac{\mathrm{d}B}{\mathrm{d}t}$
$L^{III}_{3,8}$	$X_1 = q, X_2 = tp + xq, X_3 = 2txp + (x^2 + t^2)q$	$\ddot{x} = [(\dot{x} - \dot{x}^3) + \alpha(1 - \dot{x}^2)^{3/2}$ $+\beta^2 \dot{x}(1 - \dot{x}^2)^{3/2}]t^{-1}$ $+\beta(1 - \dot{x}^2)^{5/4} t^{-1/2} \frac{\mathrm{d}B}{\mathrm{d}t}$

Remark. In theorem 7, if $I' = 0$, take $h(x) = \int^x \frac{1}{g(s)}\,\mathrm{d}s$ and if $J' = 0$, $h(x)$ is a solution of the ODE $(h'g)' + Jh' = 0$.

Theorem 8. *The SODE, corresponding to the Lie algebra $L^{I}_{3,3}$ of table 2,*

$$\ddot{x} = \alpha \mathrm{e}^{-\dot{x}} + \beta \mathrm{e}^{-\dot{x}/2} \frac{\mathrm{d}B}{\mathrm{d}t}$$

is integrable by quadratures if

$$\alpha = \frac{\beta^2}{4} - \frac{\beta^4}{2}.$$

Also, the SODE associated with the Lie algebra $L^{I,a}_{3,6}$, namely

$$\ddot{x} = \alpha \dot{x}^{\frac{a-2}{a-1}} + \beta \dot{x}^{\frac{a-3/2}{a-1}} \frac{\mathrm{d}B}{\mathrm{d}t},$$

is solvable by quadratures provided

$$\alpha = \frac{\beta^2(2a - 3)}{4(a - 1)}.$$

Proof. For $L^{I}_{3,3}$, make the change $u = \mathrm{e}^{-\dot{x}/2}$. The transformed equation reads

$$\mathrm{d}u = \left(\frac{\beta^2}{8} - \frac{\alpha}{2}\right) u^3\,\mathrm{d}t - \frac{\beta^2}{2} u^2\,\mathrm{d}B. \tag{43}$$

Using theorem 7, we find, after simple calculations that

$$I = \left(\frac{\alpha}{\beta^2} + \frac{\beta^2}{2} - \frac{1}{4}\right) u \qquad I' = \frac{\alpha}{\beta^2} + \frac{\beta^2}{2} - \frac{1}{4}$$
$$J = -\beta^2 u \qquad J' = -\beta^2 \neq 0.$$

So (43) is reducible provided $I' = 0$, i.e

$$\alpha = \frac{\beta^2}{4} - \frac{\beta^4}{2}.$$

In order to integrate (43), we follow the remark made below theorem 7 and we perform the change of variable

$$v = -\frac{2}{\beta^2}\int^u s^{-2}\,\mathrm{d}s = \frac{2}{\beta^2 u}.$$

Then (43) becomes (use Ito's formula)

$$\mathrm{d}v = \frac{2}{\beta^2}\left(\frac{\beta^2}{2} + \frac{\alpha}{\beta^2} - \frac{1}{4}\right)\frac{1}{v}\,\mathrm{d}t + \mathrm{d}B.$$

But the term within the parentheses vanishes. Thus

$$v = B + C_1$$

where C_1 is an arbitrary constant stochastic process. Recalling that

$$u = \mathrm{e}^{-\dot{x}/2} = \frac{\beta^2}{2(B + C_1)}$$

we finally obtain

$$x(t) = 2\int^t \ln\left[\frac{2}{\beta^2}(B(s) + C_1)\right] \mathrm{d}s + C_2$$

where C_2 is another arbitrary constant stochastic process. Hence (43) is integrable by quadratures.

In the case of $L^{I,a}_{3,6}$ make the transformation $u = \dot{x}$. The resulting equation is

$$\mathrm{d}u = \alpha u^{\frac{a-2}{a-1}}\,\mathrm{d}t + \beta u^{\frac{a-3/2}{a-1}}\,\mathrm{d}B. \tag{44}$$

Let

$$m = -\frac{1}{2(a-1)}.$$

Then

$$I = \left[\frac{\alpha}{\beta} - \frac{\beta}{2}(m+1)\right] u^m \qquad I' = m\left[\frac{\alpha}{\beta} - \frac{\beta}{2}(m+1)\right] u^{m-1}$$
$$J = 2m\beta u^m \qquad J' = 2m^2\beta u^{m-1}.$$

So equation (44) is reducible provided $I' = 0$ (note that J' cannot be zero since $m \neq 0$), i.e

$$\alpha = \frac{\beta^2(2a-3)}{4(a-1)}.$$

In order to integrate (44) in this case, we make the change

$$v = -\frac{1}{\beta m} u^{-m}.$$

Then, equation (44) reduces to

$$\mathrm{d}v = -(\beta m)^{-1} \left(\frac{\alpha}{\beta} - \frac{\beta}{2}(m+1) \right) v^{-1}\,\mathrm{d}t + \mathrm{d}B.$$

But the term within the parentheses vanishes. Hence

$$v = B + C_1$$

where C_1 is an arbitrary constant stochastic process. Reverting to the original variables, we obtain

$$u = \dot{x} = (-\beta m)^{-1/m} (B + C_1)^{-1/m}$$

and finally,

$$x(t) = \left(\frac{\beta}{2(a-1)} \right)^{2(a-1)} \int^t (B(s) + C_1)^{2(a-1)}\,\mathrm{d}s + C_2$$

where C_2 is another arbitrary constant stochastic process. This concludes the proof of the theorem. □

Remark. We could not integrate the canonical forms corresponding to the Lie algebras $L^b_{3,7}$, $L^I_{3,8}$, $L^{II}_{3,8}$, $L^{III}_{3,8}$, although the corresponding deterministic ODEs are integrable.

5. Symmetry breaking for scalar second-order SODEs

Since the stochastic part imposes extra constraints in the determining equations (see section 3), it is obvious that second-order scalar SODEs are less symmetric than their deterministic counterpart. Hence, a scalar second-order SODE cannot admit an n-dimensional Lie algebra with $n \geqslant 9$.

Theorem 9. *A scalar second-order SODE which admits a four-dimensional point symmetry Lie algebra is reducible to*

$$\ddot{x} = \beta \frac{\mathrm{d}B}{\mathrm{d}t} \tag{45}$$

where $\beta \neq 0$ is a constant.

Proof. Use the fact that a real four-dimensional Lie algebra contains a three-dimensional subalgebra [25, 26] and table 2. □

Example. Consider the SODE

$$\ddot{y} = \frac{(\dot{y}-1)^2}{2(y-t)} + 2\sqrt{y-t}\,\frac{\mathrm{d}B}{\mathrm{d}t}. \tag{46}$$

Using the algorithm of section 3.3 we find that its symmetries are given by

$$Y_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \qquad Y_2 = \sqrt{y-t}\,\frac{\partial}{\partial y}$$

$$Y_3 = t\frac{\partial}{\partial y} \qquad Y_4 = t\frac{\partial}{\partial t} + 3(y-t)\frac{\partial}{\partial y}.$$

It can be verified that Y_1, Y_2, Y_3 and Y_4 generate a four-dimensional Lie algebra. So according to theorem 9, equation (46) is reducible to (45) for some β. Indeed the transformation $y = x^2 + t$ reduces (46) to

$$\ddot{x} = \frac{\mathrm{d}B}{\mathrm{d}t}.$$

Remark. If a second-order scalar SODE admits a four-dimensional Lie point symmetry algebra, the transformation that maps it to (45) is just the transformation that maps its symmetries to those of (45), namely, (39). To find this transformation, one needs to solve a simple system of linear first-order partial DEs.

Next, we recall a theorem which will be crucial for the symmetry breaking of scalar second-order SODEs.

Theorem 10 (Ergorov–Turkowski). *A real Lie algebra L with $5 \leqslant \dim L \leqslant 8$ contains a four-dimensional subalgebra.*

Proof. See Turkowski [27]. □

By theorems 9 and 10 we deduce the following result.

Theorem 11 (Maximal Lie algebras). *A scalar second-order SODE can maximally admit a zero-, one-, two-, three- or four-dimensional point symmetry Lie algebra.*

6. Conclusion

In this paper, we have obtained a symmetry algorithm for SODEs. This has been applied to the study of scalar second-order SODEs. We found that a scalar SODE admitting a two-dimensional point symmetry Lie algebra with connected generators is linearizable via an invertible point transformation. We also proved that a scalar second-order SODE admitting a two-dimensional point symmetry Lie algebra is in general reducible to a homogeneous Ito equation. Furthermore, we classified scalar second-order SODEs admitting three-dimensional point symmetry Lie algebras and characterized two classes of integrable nonlinear second-order SODEs. Finally, we showed that a scalar second-order SODE can admit at most a four-dimensional point symmetry Lie algebra.

Recently, Gaeta [28] considered projectable equivalence transformations of the Ito equation (he named them 'W-symmetries'). *Stricto sensus*, these are not symmetries of the Ito equation (i.e. transformations that leave the Ito equation unchanged) but they are rather transformations that map an Ito process into another one. The approach used in this paper can be utilized to deal with non-projectable equivalence transformations of (1) and thus we can obtain an extension of the results of [28]. But this falls out of the scope of this paper, whose main focus is on Lie point symmetries.

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Appendix

Here, we give the expanded forms of the equations (21), (22) and (32), (33).

If we expand (21), (22), we obtain

$$\eta_{i,t} + \eta_{i,j} f_j - f_i(\xi_{,t} + \xi_{,j} f_j) - \xi f_{i,t} - \eta_j f_{i,j} + \tfrac{1}{2} G_{jl} G_{kl}(\eta_{i,jk} - f_i \xi_{,jk}) = 0 \tag{A1}$$

$$\xi G_{ij,t} + \eta_k G_{ij,k} + (\xi_{,k} f_i - \eta_{i,k}) G_{kj} + \frac{1}{2} G_{ij} \left[\xi_{,t} + \xi_{,k} f_k + \frac{1}{2} \xi_{,kl} G_{kp} G_{lp} + \xi_{,k} G_{kl} \frac{\mathrm{d}B_l}{\mathrm{d}t} \right] = 0. \tag{A2}$$

After expansion, (32), (33) yields

$$(\eta^{[n]} - \xi_{,t} f - \eta^{[\alpha]} \operatorname{grad}_\alpha f)|_{x^{(n)}=f} - \delta_{1n-1} G G^T \operatorname{grad} \xi = 0 \tag{A3}$$

$$\xi G_{,t} + \eta^{[\alpha]} \operatorname{grad}_\alpha G + \left(n - \frac{1}{2} \right) \frac{\mathrm{d}\xi}{\mathrm{d}t} G - (\operatorname{grad} \eta) G + (G\dot{x})^T \operatorname{grad} \xi = 0 \tag{A4}$$

where the $\eta^{[k]}$ are given by the prolongation formulae (see section 3.1) and

$$\eta^{[\alpha]} \operatorname{grad}_\alpha \psi \equiv \sum_{\alpha=0}^{\alpha=n-1} \eta_i^{[\alpha]} \psi_{,x_i^{(\alpha)}} \tag{A5}$$

for a given $\psi(t, x, \dot{x}, \ldots, x^{(n-1)})$.

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